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OPTIMAL DESIGN OF PLASTIC STRUCTURES UNDER IMPULSIVE AND DYNAMIC PRESSURE LOADING†

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Abstract—Optimal design of a rigid-plastic stepped beam and circular plate is considered in the first part of the paper assuming the mode form of motion. The form of optimal mode is sought for which a structure of constant volume attains a minimum of local or mean deflection. It is assumed that the constant kinetic energy Ko is attained by the structure through impulsive loading. Differences between optimal static and dynamic solutions are discussed. Non-uniqueness of modes is demonstrated and significance of stable mode motions is emphasized. In the second part of the paper, an optimal design of a rigid-plastic stepped beam loaded by a uniform pressure over a time interval $0 \le t \le t_1$ is considered assuming constant beam volume and looking for a design corresponding to minimum of local deflection. The solution presented is valid for moderate dynamic pressures when mode motion occurs during consecutive time intervals and no travelling plastic hinges exist.

1. INTRODUCTION

When optimizing plastic beams or plates with prescribed static loading and support conditions, the minimum of material volume or cost is sought for prescribed safety factor against plastic collapse. The optimality condition then requires constant rate of dissipation per unit length or area of the middle surface for unconstrained thickness variation or constant mean rate of dissipation for one-parameter variation of cross-section within prescribed intervals[1, 2]. Thus for the well known Foulkes failure mechanism in frame structures corresponding to optimal solution, at least one plastic hinge must occur within each prismatic member and the rate of rotation at each hinge is governed by the condition of mean rate of dissipation.

The case of dynamic loading is far more complicated since transient dynamic behaviour depends on time and space variation of loading. However, a simplified picture is obtained when considering only mode forms of motions, that is

$$\dot{u}_i(x, t) = v_i(x, t) = w_i(x)\dot{\phi}(t), \tag{1}$$

where $v_i(x, t)$ and $u_i(x, t)$ denote the velocity and the displacement of material element. It was shown by Martin and Symonds[3] that for rigid, perfectly-plastic materials, permanent mode solutions exist during the whole motion of the structure provided proper initial conditions are prescribed. Moreover, $w_i(x)$ is an eigenfunction for the case of impulsive loading, characterized by an extremum principle[4] with the corresponding eigenvalue proportional to the acceleration of motion. Usually, the initial transient motion tends to the modal form which predominates in the final period before the rest.

In the first part of the present paper, we restrict our analysis to mode motion (1) and study the optimal form of $w_i(x)$ for which a structure of constant volume attains a minimum of deflection at prescribed point or minimum of mean or maximal deflection. It is assumed that the initial instant the structure attains the kinetic energy K_0 and the subsequent motion proceeds in the modal form (1). Instead of studying general optimality criteria, we rather restrict our analysis to solution of several particular cases from which some general conclusions will be deduced. In particular, it will be shown that particular modes representing optimal solutions are

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not formed by multi-degree-of-freedom mechanism as in the static case. Moreover, the non-uniqueness of modes for some values of design parameters will be demonstrated.

In the second part of this paper, an optimal design of a rigid-plastic stepped beam loaded by a uniform pressure over a time interval $0 \le t \le t_1$ is considered. Although, the mode solution (1) is no longer valid in this case, it is assumed that different mode motions occur during consecutive time intervals. The solution for pressure loading clarifies also the important problem of stability of modes, indicating that multi-degree-of freedom modes are unstable and motion terminates usually through a one-degree-of freedom mode.

2. IMPULSIVE LOADING: OPTIMAL DESIGN OF BEAMS WITH SEGMENTWISE CONSTANT THICKNESS

Consider a rigid-plastic beam of rectangular cross-section with piecewise-constant thickness, simply supported at both edges, Fig. 1(a). At the initial instant, the kinetic energy K_0 is imparted to the beam and the subsequent motion satisfies the mode solution (1). Assuming constant volume of the beam, let us determine optimal thickness h_1, h_2, h_3 of the three portions *AB*, *BC* and *CD* that correspond to minimum of final deflection at *A*. Thus the problem is formulated as follows

$$\text{minimize} \qquad u_A \tag{2}$$

$$\text{subject to} \qquad V = 2D[ah_1 + (b-a)h_2 + (l-b)h_3],$$

$$k = \frac{1}{2}\int_0^{2l} mv^2\,\mathrm{d}x = \int_0^a \rho Dh_1 v^2(x,t)\,\mathrm{d}x + \int_a^b \rho Dh_2 v^2(x,t)\,\mathrm{d}x + \int_b^l \rho Dh_3 v^2(x,t)\,\mathrm{d}x = K_0 \tag{2'}$$

where ρ denotes the material density, $v = w(x)\phi(t)$ and $\dot{v} = w(x)\ddot{\phi}(t)$ are respectively lateral velocity and acceleleration; a, b, l, are dimensions shown in Fig. 1(a) and D denotes the beam width. Denoting by M and Q^s the bending moment and the shear force, the usual equations of motion take the form

$$\frac{\mathrm{d}M}{\mathrm{d}x} = Q^s, \qquad \frac{\mathrm{d}Q^s}{\mathrm{d}x} = \rho Dh(x)w(x)\ddot{\phi}(t). \tag{3}$$

Note that $M = M(x)$, $Q = Q(x)$ and therefore $\ddot{\phi}(t) = -q = \text{const.}$ where q denotes constant deceleration of motion; hence $v(x,t) = v_0(x) - qtw(x)$.

Assume now that the mechanism of motion is that shown in Fig. 1(b), where plastic hinges may occur at junctions of beam segments *B*, *C* and the beam center *A*. Denoting relative rotations at the hinges by θ, $\varphi - \theta$ and $\psi - \varphi$, it is evident that for positive bending moments there should be $\theta \le \varphi \le \psi$. Thus when $\theta = \varphi$, no hinge occurs at *B* and when $\psi = \varphi$ there is no

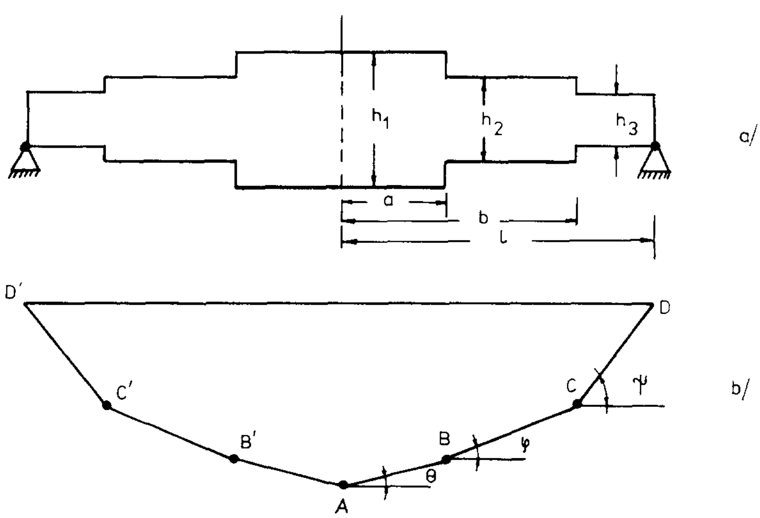


Fig. 1(a). Beam dimensions and (b) yield mechanism with plastic hinges at *A*, *B*, *C* and *B'*, *C'*.

hinge at C. For modal, motions the proportionality relations

$$\frac{\psi}{\theta}=\mu, \qquad \frac{\varphi}{\theta}=\lambda \tag{4}$$

should hold during the whole period of motion; here μ and λ are constant proportionality factors. The deflection field can now be described as follows

$$u(x,t)=\begin{cases}[(l-b)\mu+(b-a)\lambda+(a-x)]\theta(t), & \text{for } x\epsilon[0,a]\\ [(l-b)\mu-(b-x)\lambda]\theta(t), & \text{for } x\epsilon[a,b]\\ (l-x)\mu\theta(t) & \text{for } x\epsilon[b,l].\end{cases} \tag{5}$$

Let us introduce the following non-dimensional quantities

$$\alpha=\frac{a}{l},\quad \beta=\frac{b}{l},\quad \xi=\frac{x}{l},\quad \gamma=\frac{h_1}{h_3},\quad \delta=\frac{h_2}{h_3}. \tag{6}$$

The beam volume is now expressed as follows

$$V=2Dh_3l\Delta=2Dh_3l[\alpha\gamma+(\beta-\alpha)\delta+1-\beta], \tag{7}$$

and the thicknesses h_1, h_2, h_3 are

$$h_1=\frac{V\gamma}{2Dl\Delta},\qquad h_2=\frac{V\delta}{2Dl\Delta},\qquad h_3=\frac{V}{2Dl\Delta}. \tag{8}$$

Integrating the motion eqns (3) and using (5), the following expressions for bending moments at A, B, C, are obtained

$$\begin{aligned}M_A&=-\frac{1}{6}\rho Dh_3l^3\ddot{\theta}(A_1+A_2\lambda+A_3\mu),\\ M_B&=-\frac{1}{6}\rho Dh_3l^3\ddot{\theta}(B_1+B_2\lambda+B_3\mu),\\ M_C&=-\frac{1}{6}\rho Dh_3l^3\ddot{\theta}(C_1+C_2\lambda+C_3\mu).\end{aligned} \tag{9}$$

where

$$\begin{aligned}&A_1=\alpha^2\gamma(3-\alpha),\qquad A_2=[3\alpha\gamma(2-\alpha)+(\beta-\alpha)(3-2\alpha-\beta)\delta](\beta-\alpha),\\ &A_3=[3\alpha\gamma(2-\alpha)+3(\beta-\alpha)(2-\alpha-\beta)\delta+2(1-\beta)^2],\qquad B_1=3\alpha^2\gamma(1-\alpha),\\ &B_2=[6\alpha\gamma(1-\alpha)+(\beta-\alpha)(3-2\alpha-\beta)\delta](\beta-\alpha),\\ &B_3=[6\alpha\gamma(1-\alpha)+3(\beta-\alpha)(2-\alpha-\beta)\delta+2(1-\beta)^2](1-\beta),\\ &C_1=3\alpha^2\gamma(1-\beta),\qquad C_2=3(\beta-\alpha)(1-\beta)[2\gamma\alpha+(\beta-\alpha)\delta],\\ &C_3=2[3\alpha\gamma+3(\beta-\alpha)\delta+1-\beta](1-\beta)^2.\end{aligned} \tag{10}$$

The bending moments M_A, M_B, M_C must satisfy the following inequalities

$$M_A\le M_A{}^0=\frac{1}{4}D\sigma_0h_1{}^2,\qquad M_B\le M_B{}^0=\frac{1}{4}D\sigma_0h_2{}^2,\qquad M_C\le M_C{}^0=\frac{1}{4}D\sigma_0h_3{}^2. \tag{11}$$

Introducing the quantities

$$N=\frac{3\sigma_0V}{4\rho l^4D},\qquad P=-\frac{\ddot{\theta}}{N},\qquad Q=-\frac{\ddot{\varphi}}{N},\qquad R=-\frac{\ddot{\psi}}{N}, \tag{12}$$

the inequalities (11) can be presented in the form

$$
\begin{aligned}
X &= A_1P + A_2Q + A_3R - \frac{\gamma^2}{\Delta} \le 0, \\
Y &= B_1P + B_2Q + B_3R - \frac{\delta^2}{\Delta} \le 0, \\
Z &= C_1P + C_2Q + C_3R - \frac{1}{\Delta} \le 0.
\end{aligned} \tag{13}
$$

Note that $R \ge Q \ge P$ since otherwise the dissipation rate at respective hinges would be negative. When any of inequalities (13) is satisfied as strong inequality, the corresponding cross-section remains rigid and plastic rotation occurs when (13) is satisfied as equality. Note that when (11) or (13) are satisfied, the bending moments nowhere exceed the yield moments since the maximum of $M(x)$ is attained only at $x = 0$.

Now, let us derive the formula for final deflection at the beam center. Starting from the representation (5) and noting that $\ddot{\theta} = \text{const.}$, the beam velocity is expressed as follows

$$v = w(x)[\ddot{\theta}t + \dot{\theta}(0)]$$

and the motion ends at the time

$$t_f = -\frac{\dot{\theta}(0)}{\ddot{\theta}}. \tag{14}$$

Thus the final deflection at the center $\zeta = 0$ equals

$$u(0, t_f) = -\frac{1}{2} l[(1-\beta)\mu + (\beta - \alpha)\lambda + \alpha] \frac{\dot{\theta}^2(0)}{\ddot{\theta}}. \tag{15}$$

In order to calculate the ratio $\dot{\theta}^2(0)/\ddot{\theta}$ occuring in (15), let us apply the energy balance equation $\dot{K} + \dot{A} = 0$, where $K(t)$ denotes the total kinetic energy and $A(t)$ is the plastic work dissipated in hinges A, B, C and A', B', C'. Since

$$v(x, t) = w(x)\dot{\theta}(t) = v(x, 0)\frac{\dot{\theta}(t)}{\dot{\theta}(0)} \tag{16}$$

the rate of kinetic energy can be expressed as follows

$$
\begin{aligned}
\dot{K} &= 2D\rho \int_0^l \frac{1}{2} v^2(x, t)h(x)\,\mathrm{d}x = 2D\rho \left[\frac{\dot{\theta}^2(t)}{\dot{\theta}^2(0)}\right]^{\cdot} \int_0^l \frac{1}{2} v^2(x, 0)h(x)\,\mathrm{d}x \\
&= \frac{2K_0}{\dot{\theta}^2(0)} \dot{\theta}(t)\ddot{\theta}
\end{aligned} \tag{17}
$$

and the rate of plastic work is

$$\dot{A} = 2[M_A\dot{\theta} + M_B(\dot{\varphi} - \dot{\theta}) + M_c(\dot{\psi} - \dot{\varphi})] = \frac{D\sigma_0 h_3^2}{4}[\gamma^2 + (\lambda - 1)\delta^2 + \mu - \lambda]\dot{\theta}(t). \tag{18}$$

From (17) and (18) one obtains

$$\frac{\dot{\theta}^2(0)}{-\ddot{\theta}} = \frac{8K_0}{D\sigma_0 h_3^2[\gamma^2 + (\lambda - 1)\delta^2 + \mu - \lambda]}. \tag{19}$$

Substituting (19) into (15) and using (8), (12), we finally obtain

$$u(0, t_f) = \frac{16 l^3 K_0}{\sigma_0 V^2} F(\alpha, \beta, \gamma, \mu), \tag{20}$$

where

$$F(\alpha, \beta, \gamma, \mu) = \frac{[(1-\beta)R + (\beta-\alpha)Q + \alpha P]\Delta^2}{R + (\delta^2 - 1)Q + (\gamma^2 - \delta^2)P} \tag{21}$$

The problem of optimal design can now be solved as follows. For any combination of parameters $0 \le \alpha \le \beta \le 1$ and $1 \le \delta \le \gamma$ the proper yield condition can be selected from Table 1, the quantities P, Q, R calculated from (13) and the value of F from (21).

Table 1 shows all fundamental yield mechanisms with corresponding inequalities for bending moments and conditions for P, Q, R. Using this table, some special cases can easily be discussed.

(i) For a beam of constant thickness there is $\gamma = \delta = 1$ and only the mode 1 from Table 1 is possible. Now we have $P = Q = R = 0.5$ and the relation (20) provides $F = 1$.

(ii) When $\alpha = \beta$, only the portions of thicknesses h_1 and h_3 occur. The second inequality (13) is now not applicable since plastic hinges at B and C coincide and $A_2 = C_2 = 0$. Thus (13) takes the form

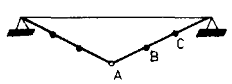
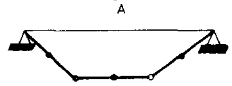
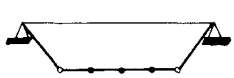
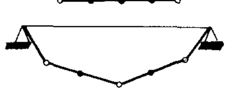
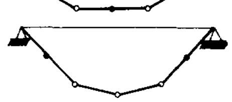
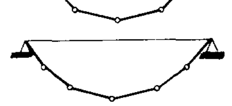
$$\begin{aligned} X &= A_1P + A_3R - \frac{\gamma^2}{\Delta} \le 0, \\ Z &= C_1P + C_3R - \frac{1}{\Delta} \le 0. \end{aligned} \tag{23}$$

The three fundamental yield mechanisms representing eigenfunctions are shown in Table 2.

3. IMPULSIVE LOADING ON BEAMS: DISCUSSION OF NUMERICAL RESULTS

(i) Let us begin our discussion with the simpler case when $\alpha = \beta$ that is with the beam with two steps. Figures 2–4 show the results of numerical calculations. Figure 2 shows the regions of occurence of the three modes of Table 2 in the plane (α, γ). It is seen that the mode 3 with two active hinges at A and C may occur only in a narrow dashed zone. Moreover, as it follows from Fig. 3, all three modes are possible in this region. In fact, for any value of α, the mode 1 with the central hinge may coexist with two other modes within the interval $\gamma_* \le \gamma \le \gamma^*$. This

Table 1. Mode forms for the three-segment beam and inequalities for bending moments

Case	Conditions for bending moments	Conditions for P, Q, R
1. 	$M_A = M_{AS}$ $M_B \le M_{BS}$ $M_C \le M_{CS}$	$P = Q = R > 0$ $X = 0,\ Y \le 0,\ Z \le 0$
2. 	$M_A \le M_{AS}$ $M_B = M_{BS}$ $M_C \le M_{CS}$	$P = 0,\ Q = R > 0$ $X \le 0,\ Y = 0,\ Z \le 0$
3. 	$M_A \le M_{AS}$ $M_B \le M_{BS}$ $M_C = M_{CS}$	$P = Q = 0,\ R > 0$ $X \le 0,\ Y \le 0,\ Z = 0$
4. 	$M_A = M_{AS}$ $M_B \le M_{BS}$ $M_C = M_{CS}$	$R > P = Q > 0$ $X = 0,\ Y \le 0,\ Z = 0$
5. 	$M_A \le M_{AS}$ $M_B = M_{BS}$ $M_C = M_{CS}$	$P = 0,\ R > Q > 0$ $X \le 0,\ Y = Z = 0$
6. 	$M_A = M_{AS}$ $M_B = M_{BS}$ $M_C \le M_{CS}$	$R = Q > P > 0$ $X = Y = 0,\ Z \le 0$
7. 	$M_A = M_{AS}$ $M_B = M_{BS}$ $M_C = M_{CS}$	$R > Q > P > 0$ $X = Y = Z = 0$

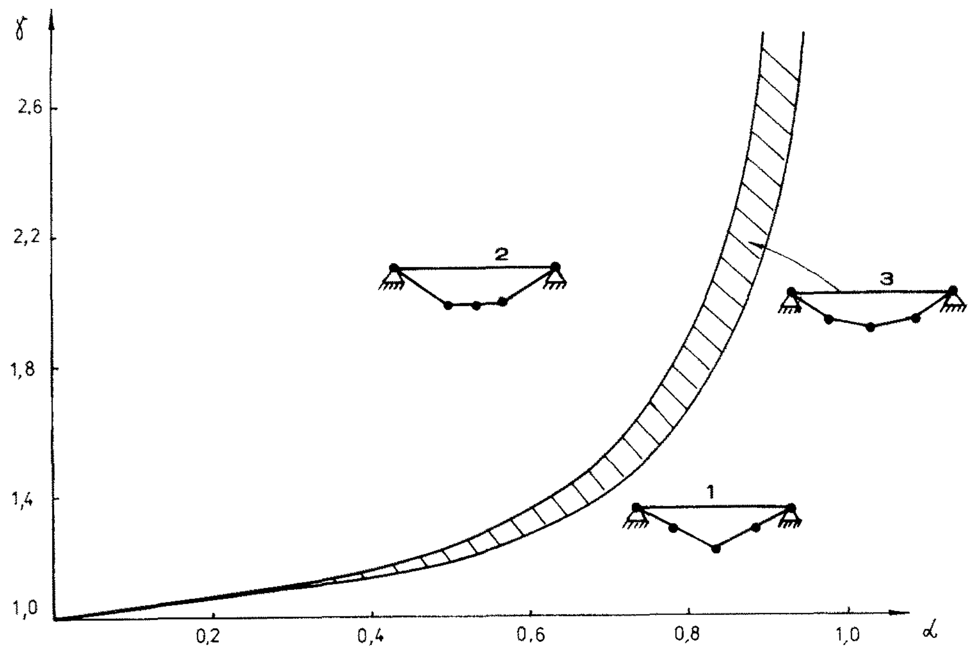


Fig. 2. Two-segment beam: regions of occurence of modes 1, 2 and 3 in the plane (α, γ).

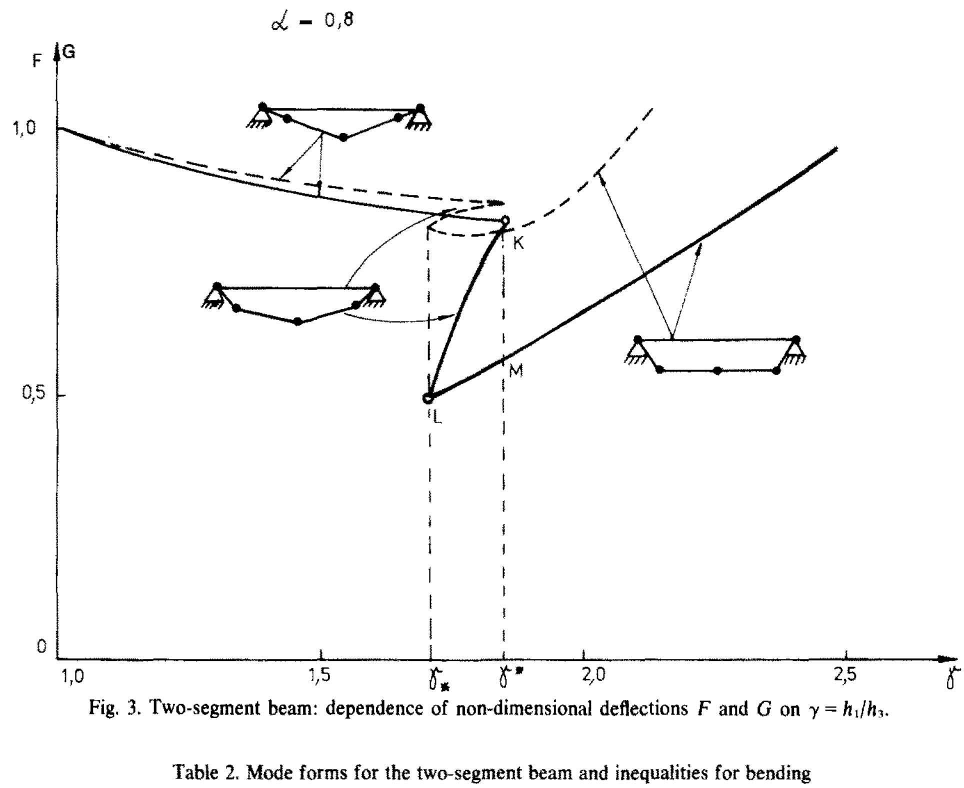
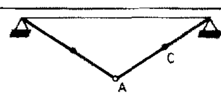
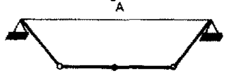


Fig. 3. Two-segment beam: dependence of non-dimensional deflections F and G on $\gamma = h_1/h_3$.

Table 2. Mode forms for the two-segment beam and inequalities for bending moments

Case		Conditions for bending moments	Conditions for P, R
1.		$M_A = M_{AS}$ $M_C \leq M_{CS}$	$P = R > 0$ $X = 0,\ Z \leq 0$
2.		$M_A \leq M_{AS}$ $M_C = M_{CS}$	$P = 0,\ R > 0$ $X \leq 0,\ Z = 0$
3.		$M_A = M_{AS}$ $M_C = M_{CS}$	$R > P > 0$ $X = Z = 0$

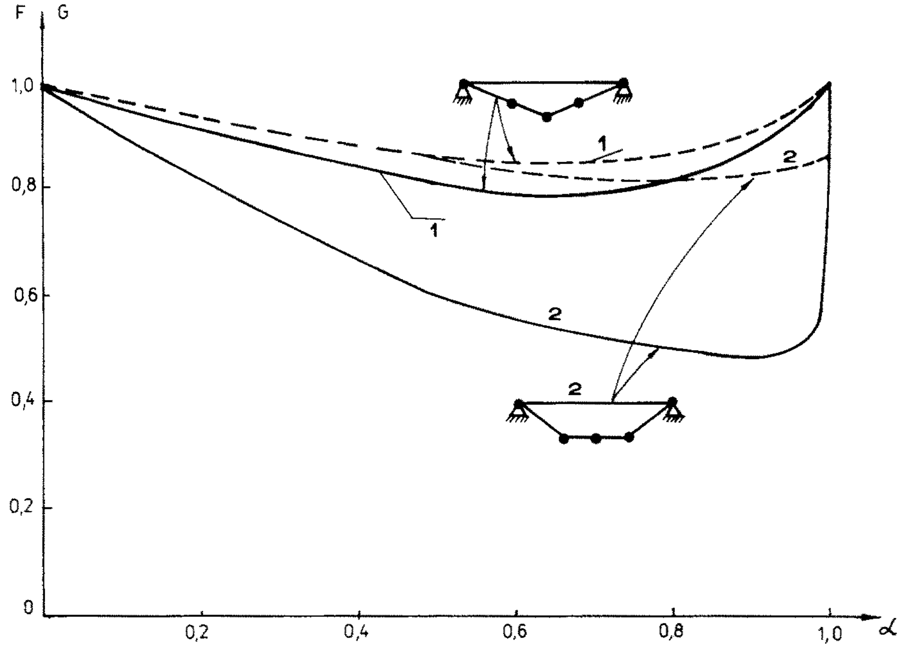


Fig. 4. Two-segment beam: dependence of minimal values of F and G on $\alpha = a/l$ for modes 1 and 2.

interval may be called the domain of instability of modes since abrupt change from one mode to another may occur for infinitesimal changes of parameters α or γ.

Figure 4 shows the dependence of minimal values of F corresponding to points K and L on the parameter α for modes 1 and 2. It is evident that the mode 2 with a rigid central portion corresponds to smallest final deflection and this solution is preferable. The theoretical minimal value of F equals $F_{min} = 0.48$ and is attained for $\alpha = 0.89$, $\gamma = 2.2$. This solution corresponds to point L in Fig. 3. Moreover, in order to avoid designs within instability domain, a solution represented by point M can be selected; this would provide parameters $\alpha = 0.9$, $\gamma = 2.6$ and $F = 0.06$. As compared to the beam of constant thickness, the deflection at the center would be reduced by 40%.

Instead of considering the local deflection, we could also provide a design minimizing the weighted mean deflection

$$\bar{u}(t_f) = \left[\int_0^l u^2(t_f, x)h(x)\,dx\right]^{1/2}. \tag{24}$$

For simplicity, let us calculate the expression for $\bar{u}(t_f)$ in the case $\alpha = \beta$.

In view of (5), we obtain

$$\bar{u}(t_f) = l\frac{\sqrt{(lh_3)}}{3}\{(1-\alpha)^2(3\alpha\gamma + 1 - \alpha)\lambda^2 + \alpha^2\gamma[3(1-\alpha)\lambda + \alpha]\}^{1/2}\theta(t_f). \tag{25}$$

Introducing again the functions P and R defined by (12), it is found that the minimum of weighted mean deflection leads to minimization of the following factor

$$G = \frac{\Delta^{3/2}}{(\gamma^2-1)P + R}\{(1-\alpha)^2(3\alpha\gamma + 1 - \alpha)R^2 + \alpha^2\gamma[3(1-\alpha)PR + \alpha P^2]\}^{1/2}. \tag{26}$$

The values of G are plotted in Figs. 3 and 4 by dotted lines. It is seen that jumps in mean deflection are smaller in the instability domain than previously although all previous conclusions remain valid. The global minimum of G now occurs at $\alpha = 0.77$, $\gamma = 1.6$ and $G_{min} = 0.81$; thus the optimal solution provides the reduction in mean deflection equal to 19%.

(ii) Let us now discuss the case $h_1 \neq h_2 \neq h_3$. This case is much more complicated since all seven modes of Table 1 should be investigated; moreover, the topology of instability domains is much more complicated. Figure 5 shows regions where different modes occur in the plane (α, γ) for $\beta = 0.9$ and $\delta = 1.4$. It is seen that six or all seven modes may exist for some values of design parameters.

Figure 6 illustrates the dependence of F on α for $\beta = 0.9$, $\gamma = 2.4$ and $\delta = 1.4$, and variation of modes with varying α. In order to find a global minimum of F, the calculations were carried out for all possible values of design parameters, assuming the mesh $\Delta\alpha = \Delta\beta = \Delta\delta = 0.05$. It was found that $F_{min} = 0.45$ occurs for $\alpha = 0.60$, $\beta = 0.90$, $\gamma = 2.40$, $\delta = 1.90$ for which the motion corresponds to mode 3. When comparing this value with the optimal solution for a two-step beam ($F_{min} = 0.48$), it can be concluded that the latter solution corresponds practically to optimal design. Thus there is no need to use multi-step designs in the case of impulsive loading.

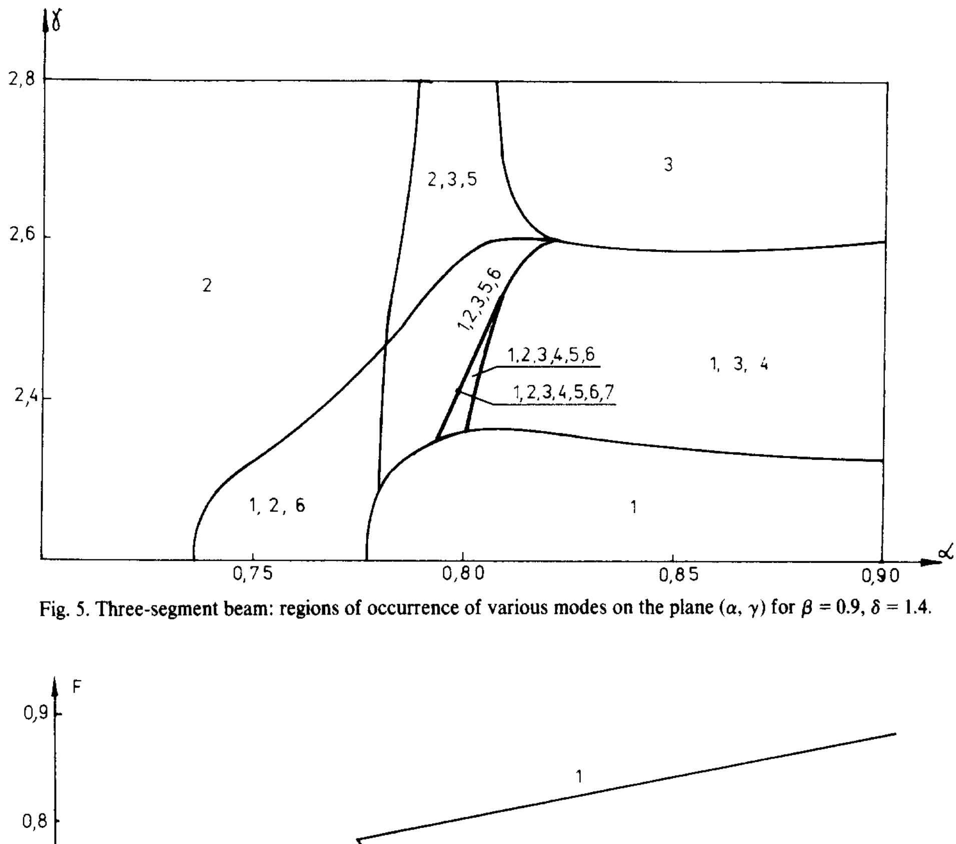


Fig. 5. Three-segment beam: regions of occurrence of various modes on the plane (α, γ) for $\beta = 0.9$, $\delta = 1.4$.

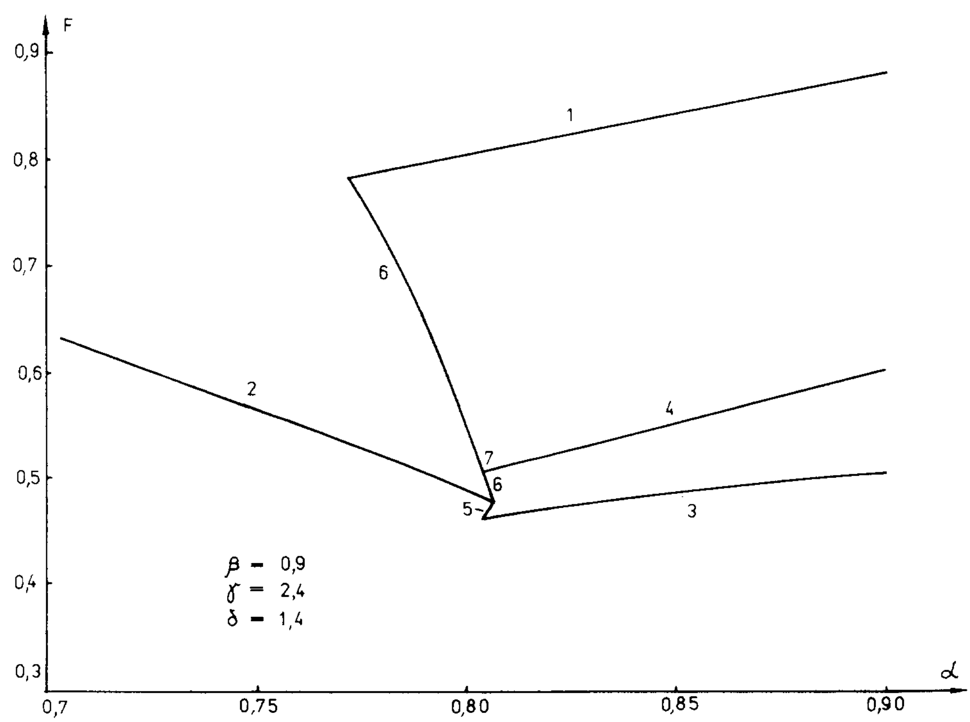


Fig. 6. Three-segment beam: dependence of non-dimensional deflection F on α for $\beta = 0.9$, $\gamma = 2.4$ and $\delta = 1.4$.

4. CIRCULAR PLATE UNDER IMPULSIVE LOADING

Following the analysis for the beam, let us consider now a circular plate with segmentwise constant thickness, simply supported at the outer edge. For simplicity, consider only two portions $0 \le r \le a$ and $a \le r \le R$ of thicknesses h_1 and h_2. The equations of motion take the form

$$\frac{d}{dr}(rM_r) - M_\varphi = rQ^s, \qquad \frac{d(rQ^s)}{dr} = r\rho\ddot{u}, \tag{27}$$

where M_r, M_φ are bending moments and Q^s denotes the shear force. The rates of curvatures are

$$\dot{k}_r = -\frac{d^2\dot{u}}{dr^2}, \qquad \dot{k}_\varphi = -\frac{1}{r}\frac{d\dot{u}}{dr} \tag{28}$$

Assuming the Tresca yield condition and the stress régime $M_\varphi - M_0 = 0$, $0 \le M_r \le M_\varphi$, the deflection field can be presented in the form

$$u(r,t) = \begin{cases} (R-a)\psi(t) + (a-r)\theta(t), & 0 \le r \le a, \\ (R-r)\psi(t), & a \le r \le R, \end{cases} \tag{29}$$

where θ and ψ are the angles of rotation at hinge lines A and C. It follows from (28) that $\dot{k}_r = 0$, $\dot{k}_\varphi = \dot{\theta}/r$ and $\dot{k}_r = 0$, $\dot{k}_\varphi = \dot{\psi}/r$ within the central and outer portions. In order to have $\dot{k}_\varphi > 0$, there must be $\dot{\theta} \ge 0$ and $\dot{\psi} \ge 0$.

Since for mode solution (1) the static field does not depend on time, both $\ddot{\theta}$ and $\ddot{\psi}$ are constant. Thus the eqns (27) can easily be integrated. When both hinges at $r = 0$ and $r = a$ are active, the stress profile is that shown in Fig. 7(a) and we obtain

$$M_r = M_{01} + \frac{1}{12}\rho h_1 r^2[2(R-a)\ddot{\psi} + (2a-r)\ddot{\theta}], \qquad 0 \le r \le a,$$

$$rM_r = aM_{01} + (r-a)M_{02} + \frac{1}{12}\rho a^2 h_1[2(R-a)(3r-2a)\ddot{\psi} + a(2r-a)\ddot{\theta}]$$

$$+\frac{1}{12}\rho h_2[2R(r^3 - 3a^2r + 2a^3) - (r^4 - 4a^3r + 3a^4)]\ddot{\psi}, \qquad a \le r \le R \tag{30}$$

where $M_{01} = (1/4)\sigma_0 h_1^2$ and $M_{02} = (1/4)\sigma_0 h_2^2$ are bending yield moments.

Introduce the non-dimensional quantities

$$\alpha = \frac{a}{R}, \qquad \gamma = \frac{h_1}{h_2}, \qquad N = \frac{3\sigma_0 V}{\pi\rho R^5}, \qquad S = \frac{\ddot{\theta}}{N}, \qquad T = \frac{\ddot{\psi}}{N}. \tag{31}$$

From the condition of constant volume, $V = \pi R^2 h_2 + \pi a^2(h_1 - h_2)$ we obtain

$$h_1 = \frac{V\gamma}{\pi R^2 \Delta}, \qquad h_2 = \frac{V}{\pi R^2 \Delta} \tag{32}$$

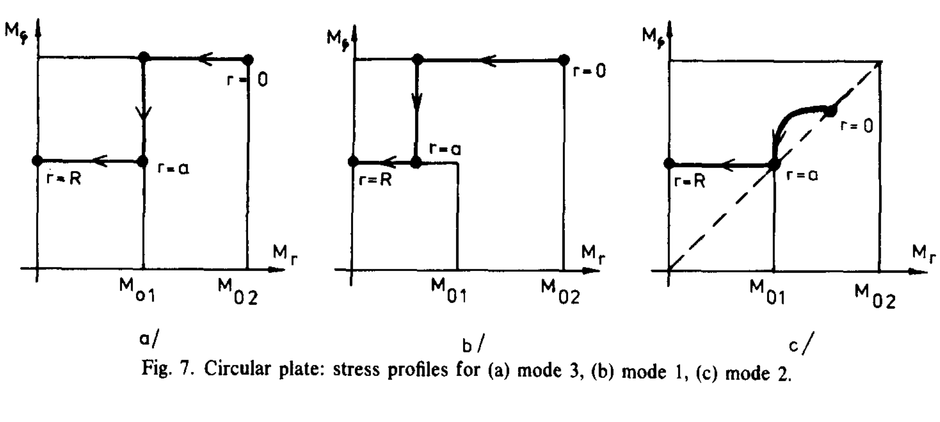


Fig. 7. Circular plate: stress profiles for (a) mode 3, (b) mode 1, (c) mode 2.

where $\Delta = 1-\alpha^2+\alpha^2\gamma$. Satisfying the boundary conditions $M_r(a) = M_{02}$, $M_r(R) = 0$, we obtain the following set of equations

$$A_1S + A_2T = -\frac{\gamma^2-1}{\Delta},$$

$$B_1S + B_2T = -\frac{\alpha\gamma^2+1-\alpha}{\Delta}, \tag{33}$$

where

$$A_1 = \alpha^3\gamma, \qquad A_2 = 2\alpha^2\gamma(1-\alpha), \qquad B_1 = \alpha^3\gamma(2-\alpha),$$

$$B_2 = 2\alpha^2\gamma(1-\alpha)(3-2\alpha)+(1-\alpha)^3(1+3\alpha). \tag{34}$$

The formula for final deflection at the plate center can be derived similarly as previously. Denoting $\lambda = \psi/\theta = \dot{\psi}/\dot{\theta} = \ddot{\psi}/\ddot{\theta}$, we can easily find from (29) that

$$u(0,t_f) = -\frac{\dot{\theta}_0^2R}{2\ddot{\theta}}[(1-\alpha)\lambda+\alpha], \tag{35}$$

and the term $\dot{\theta}_0^2/\ddot{\theta}$ can be found from the energy equation $\dot{A}+\dot{K} = 0$. The rate of kinetic energy now equals

$$\dot{K} = K_0\frac{2\dot{\theta}\ddot{\theta}}{\dot{\theta}_0^2} \tag{36}$$

and the dissipation rate can be expressed as follows

$$\dot{A} = 2\pi\int_0^a M_{01}\dot{\theta}\,\mathrm{d}r + 2\pi\int_a^R M_{02}\dot{\psi}\,\mathrm{d}r - 2\pi\int_{a-0}^{a+0} M_r\frac{\mathrm{d}^2\dot{u}}{\mathrm{d}r^2}r\,\mathrm{d}r$$

$$= 2\pi R[(M_{01}-M_{02})\alpha\dot{\theta}+M_{02}\dot{\psi}]. \tag{37}$$

Using (36) and (37), from the equation $\dot{A} = -\dot{K}$, one obtains

$$\theta(t_f) = -\frac{\dot{\theta}_0^2}{2\ddot{\theta}} = \frac{2K_0\pi R^3\Delta^2}{\sigma_0V^2[(\gamma^2-1)\alpha+\lambda]} \tag{38}$$

and the function to be minimized is

$$F = \Delta^2\frac{\alpha S+(1-\alpha)T}{\alpha S(\gamma^2-1)+T} \tag{39}$$

subject to the condition $T \le S \le 0$.

In the case when the hinge at $r = a$ is inactive (Fig. 7b), we have $\theta = \psi$, $S = T$ and

$$F = \frac{\Delta^2}{\alpha\gamma^2+1-\alpha}. \tag{40}$$

Similarly, when the central zone $0 \le r \le a$ is rigid, (Fig. 7c), there is $\theta = 0$, $S = 0$ and (39) gives

$$F = (1-\alpha)\Delta^2. \tag{41}$$

When instead of minimizing the central deflection we want to determine a design corresponding to minimum of weighted mean deflection $\bar{u}(t_f)$, that is

$$\bar{u}(t_f) = \left(\pi\int_0^R w^2hr\,\mathrm{d}r\right)^{1/2} = R\sqrt{\left(\frac{V}{12\Delta}\right)}\{[6\alpha^2\gamma(1-\alpha)^2+(1-\alpha)^3(1+3\alpha)]\lambda^2$$

$$+4\alpha^3\gamma(1-\alpha)\lambda+\alpha^4\gamma\}^{1/2}\theta(t_f) \tag{42}$$

the function to be minimized is

$$H = \frac{\Delta^{3/2}}{\alpha(\gamma^2-1)S+T}\{\alpha^3\gamma S[\alpha S+4(1-\alpha)T]+(1-\alpha)^2[6\alpha^2\gamma+(1-\alpha)(1+3\alpha)]T^2\}^{1/2}. \quad (43)$$

Note that for a plate of constant thickness we have $\gamma = 1$, $S = T$ and $F = H = 1$.

The results of calculations are shown in Figs. 8–10. It is seen that all features of the solution for a beam are preserved in this case. Figure 8 illustrates the ranges of α and γ for which the

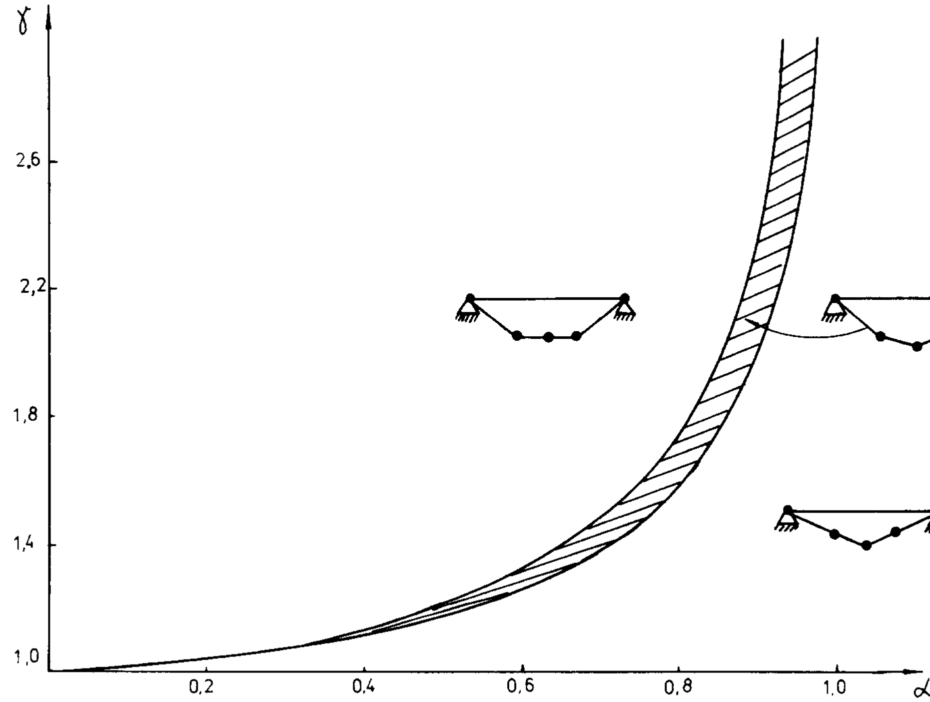


Fig. 8. Circular plate: regions of occurence of modes 1, 2 and 3 in the plane (α, γ).

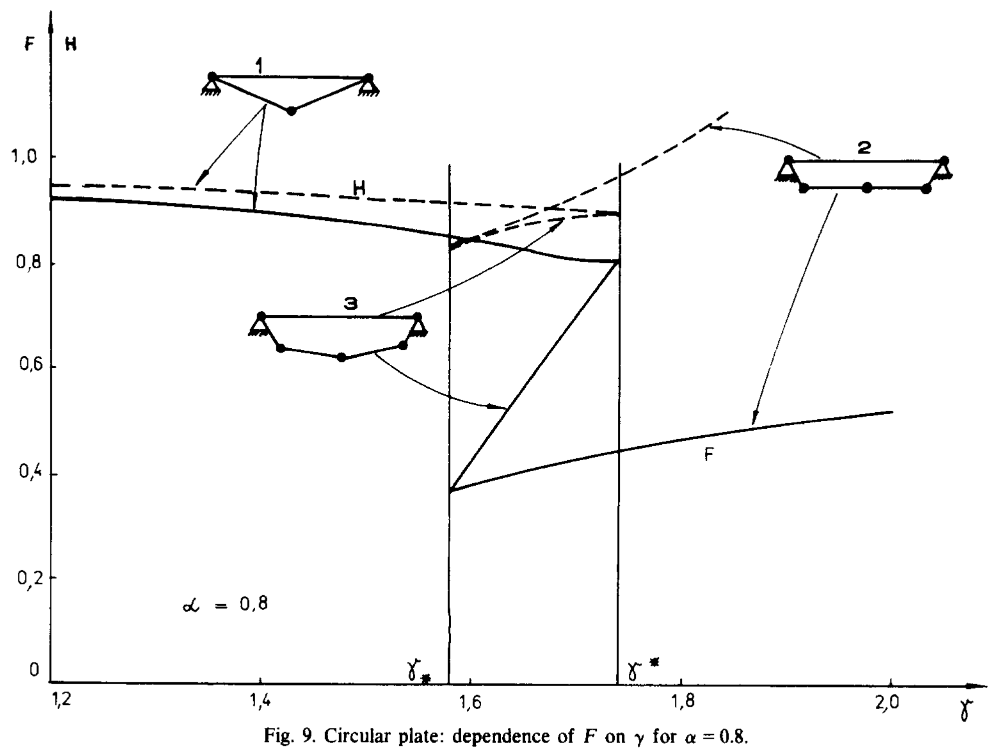


Fig. 9. Circular plate: dependence of F on γ for $\alpha = 0.8$.

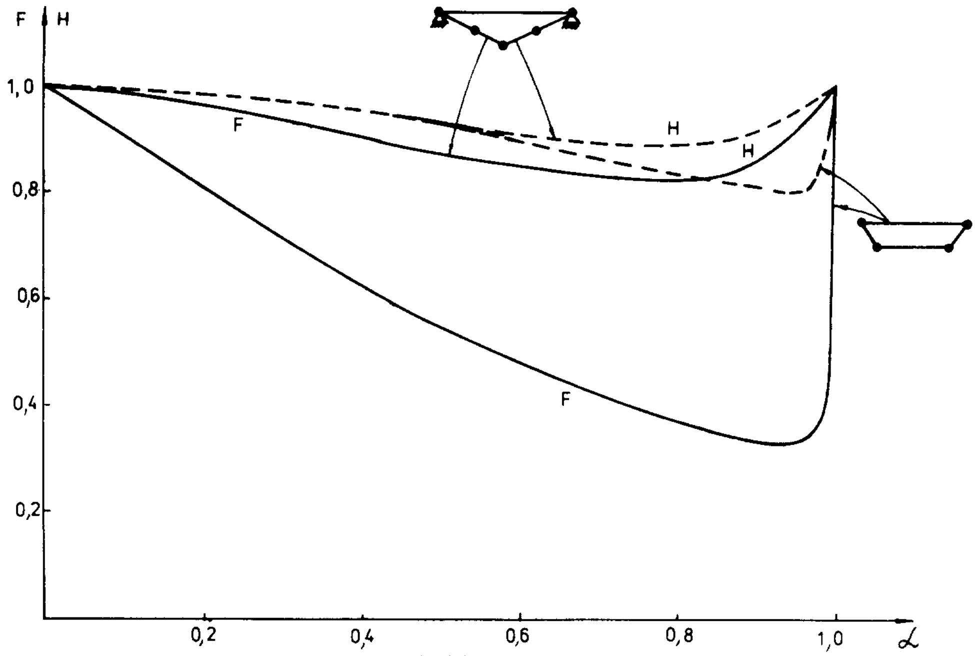


Fig. 10. Circular plate: dependence of minimal values of F and H on α for modes 1 and 2.

three modes occur: again the mode 3 may occur only in a narrow dashed zone. Figure 9 shows the dependence of F on γ for $\alpha = 0.8$. It is seen that for $\gamma_* \leq \gamma \leq \gamma^*$ the three modes may occur simultaneously and minimum of F takes place for $\gamma = \gamma_*$. Figure 10 shows the dependence of minimum values of F on α. It is found that minimum of F occurs for $\alpha = 0.95$ $\gamma = 2.75$ and $F_{MIN} = 0.33$. Dotted lines in Figs. 9 and 10 show the dependence of H on γ and α. In both cases minimum design corresponds to the modal form with a rigid central portion and plastic hinge near the support.

5. IMPULSIVE LOADING: DISCUSSION OF RESULTS

The presented examples indicate that in minimizing local or mean deflections under impulsive loading, the concept of a simultaneous failure mechanism widely used in the static case cannot directly be extended to dynamic loading. In fact, a minimum of local or mean deflection corresponds to a mode represented by a one degree-of freedom failure mechanism with plastic hinge near the support and the central portion undergoing a rigid body motion. This result may be interpreted by considering initial velocity field and the dissipation rate within the structure. For given total initial kinetic energy K_0 the corresponding velocity field will possess a minimal value of maximal velocity for a uniform velocity field which represents a rigid body motion. If constraints are imposed by supports, this uniformity may occur only within some portions of the structure. On the other hand, the maximum dissipation capacity will be attained for uniform dissipation rate within the structure and plastic flow should occur within all structural members. The compromise between these two requirements leads to designs of stepped beams and plates obtained in this work. It is also clear that requirement of minimum of local, or weighted mean deflection leads to different designs and this dependence was illustrated in this paper.

Non-uniqueness and instability of some modes is also an interesting feature of the solutions discussed here. Although stability of modes was not investigated, the analysis of the case of pressure loading indicates that multi-degree-of-freedom modes are unstable and the motion terminates through a one-degree-of freedom mode. The question of stability of modes is also discussed in a paper by Symonds and Wierzbicki[5] concerned with mode motions in plastic and viscoplastic structures.

6. OPTIMAL DESIGN OF PLASTIC BEAMS UNDER DYNAMIC PRESSURE LOADING

In this section we shall investigate the related problem of optimal design in the case of pressure loading which is uniform over the beam and constant during the time interval

$0 \le t \le t_1$. Consider the simplest case when the beam consists of two portions AB and AC of thicknesses h_1 and h_2 and the width D, Fig. 11(a). The yield mechanism is shown in Fig. 11(b) with plastic hinges at A and B and angles of rotation θ and $\psi - \theta$ respectively. We shall investigate the beam motion and the effect of varying ratios $\gamma = h_1/h_2$ and $\alpha = a/h$ on the final deflection at A. It was found for the case of impulsive loading that the one-step design such as that of Fig. 11(a) closely approaches the optimal solution and there is no need for a multi-step design. It can therefore be expected that optimizing the design of Fig. 11(a) under pressure loading will provide a solution close to the theoretical optimum.

6.1. *Theoretical analysis of beam motion*

Whereas in the case of impulsive loading we assumed the mode solution to persist during the whole motion, the present case requires consideration of particular cases of motion with different ratios of rotation rates at plastic hinges A and B. We will not, however, consider the case when hinges may move within the portions AB and BC which takes place for high values of pressure p. Thus the present case can be referred to as "moderate pressure solution" and its validity will further be examined.

$$\dot{u}(x, t) = \begin{cases} (l - a)\dot{\psi}(t) + (a - x)\dot{\theta}(t), & 0 \le x \le a, \\ (l - x)\dot{\psi}(t), & a \le x \le l, \end{cases} \tag{44}$$

where $\dot{\theta} = \dot{\theta}(t)$ and $\dot{\psi} = \dot{\psi}(t)$ are rates of rotation of portions AB and BC. Denoting by M and Q^s the bending moment and the shear force, the equations of motion take the form

$$\frac{dM}{dx} = Q^s, \qquad \frac{dQ^s}{dx} = -p + \rho D h \ddot{u} \tag{45}$$

where D denotes the beam width. Since p is constant in time for $0 \le t \le t_1$, the bending moments do not depend on time when $\ddot{u} = \ddot{u}(x)$ that is $\ddot{\psi} = \text{const.}$ and $\ddot{\theta} = \text{const.}$

The optimization problem can be formulated as follows: for a beam of constant volume $V = 2Dh_2l[\alpha\gamma + 1 - \alpha]$ we want to find such values of α and γ for which the final deflection at the beam center is minimum. Instead of looking for the proper optimality condition, we determine the optimal solution through the numerical search analysis.

Integrating the equations of motion (45) and satisfying the boundary conditions for $x = 0$ and $x = l$, the expressions for bending moments can be found and moments at A and B can be calculated. Since for moderate pressures, the bending moments attain only extremum at $x = 0$, it is obvious that plastic hinges may occur at A and B and the following inequalities must be satisfied

$$M_A \le M_{0A} = \frac{1}{4} D\sigma_0 h_1^2, \qquad M_B \le M_{0B} = \frac{1}{4} D\sigma_0 h_2^2. \tag{46}$$

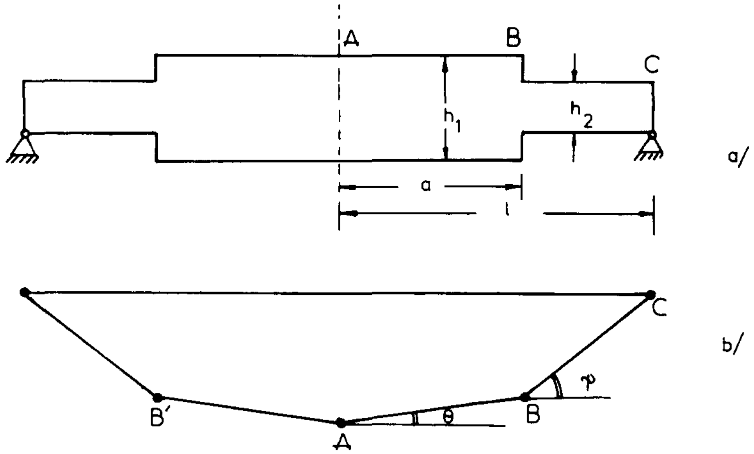


Fig. 11(a). Beam dimensions and (b) yield mechanism.

Introducing the non-dimensional quantities

$$\alpha = \frac{a}{l}, \quad \gamma = \frac{h_1}{h_2}, \quad N = \frac{3\sigma_0 V}{4\rho l^4 D}, \quad P = \frac{6p}{N\rho V}, \quad S = \frac{\ddot{\theta}}{N}, \quad T = \frac{\ddot{\psi}}{N}, \tag{47}$$
$$\Delta = 1 - \alpha + \alpha\gamma,$$

the inequalities (46) can be expressed as follows

$$-\alpha^2\gamma(3-\alpha)S - (1-\alpha)[3\alpha\gamma(2-\alpha) + 2(1-\alpha)^2]T \le \frac{\gamma^2}{\Delta} - P\Delta, \tag{48'}$$

$$-3\alpha^2\gamma S - 2(1-\alpha)(3\alpha\gamma + 1 - \alpha)T \le \frac{1}{(1-\alpha)\Delta} - P(1+\alpha)\Delta. \tag{48''}$$

Some particular cases of general yield mechanism shown in Fig. 11(b) can be discussed in view of (48). Assume first that the plastic hinge occurs only at A; then $S = T$ and $\dot{\theta} = \dot{\psi}$ and the weak inequality (48′) is satisfied as equality. Similarly, when the plastic hinge occurs at B, we have $S = 0$, $\dot{\theta} = 0$ and the second weak inequality (48″) is satisfied as equality. Finally, when the two plastic hinges are active at A and B, the conditions (48′) and (48″) are satisfied as equalities which should be used in order to determine S and T. It turns out that during the consecutive phases of motion these particular mechanisms occur in different order. Let us call them respectively the mechanisms 1, 2, and 3. Here, several particular cases of motion may exist. Let us discuss them in more detail.

Case (1–1). This case occurs when the mechanism 1 with the central hinge occurs during the time interval $0 \le t \le t_1$ when pressure acts on the beam and for $t_1 \le t \le t_f$ when the free motion of beam occurs until final rest. At the end of the first period, we have

$$\theta(t_1) = \psi(t_1) = \frac{1}{2} NS_1 t_1^2, \qquad w(t_1) = \frac{1}{2} Nlt_1^2 S_1, \tag{49}$$

where S_1 is determined from the equality (48′). In the second phase, the angular velocity $\dot{\theta}(t)$ can be found from the equation

$$\dot{\theta}(t) = \dot{\psi}(t) = \dot{\theta}(t_1) + N\bar{S}_1(t - t_1) = N[S_1 t_1 + \bar{S}_1(t - t_1)] \tag{50}$$

where $\bar{S}_1$ is determined from (48′) with $P = 0$. The motion ends at $t = t_f$ when $\dot{\theta}(t_f) = 0$, hence

$$t_f = t_1\left(1 - \frac{S_1}{\bar{S}_1}\right). \tag{51}$$

Now the total rotation and the permanent final deflection at A can be expressed in the form

$$\theta(t_f) = \theta(t_1) + \dot{\theta}(t_1)(t_f - t_1) + \frac{1}{2} N\bar{S}_1(t_f - t_1)^2 = \frac{1}{2} Nt_1^2 S_1\left(1 - \frac{S_1}{\bar{S}_1}\right) \tag{52}$$

$$u(0, t_f) = \frac{1}{2} Nlt_1 W_1 = \frac{1}{2} Nlt_1^2 S_1\left(1 - \frac{S_1}{\bar{S}_1}\right). \tag{53}$$

Case 1 occurs when the condition (48″) is satisfied as strong inequality, both for $P \ne 0$ and $P = 0$. This inequality must therefore be satisfied for $S = S_1 > 0$, $T_1 = S_1$, $P \ne 0$ and $S = \bar{S}_1 < 0$, $T = \bar{S}_1$, $P = 0$.

Case 2 (1–3–2). This case corresponds to the sequence of mechanisms 1, 3 and 2. The first phase $0 \le t \le t_1$ corresponding to the mechanism 1 is described by eqns (49). During the second phase, we have

$$\dot{\theta}(t) = N[S_1 t_1 + \bar{S}_3(t - t_1)], \qquad \dot{\psi}(t) = N[S_1 t_1 + \bar{T}_3(t - t_1)] \tag{54}$$

where $\bar{S}_3$ and $\bar{T}_3$ are determined from equalities (48′) and (48″) for $P = 0$. From the condition $\dot{\psi} > \dot{\theta}$ it follows that $\bar{T}_3 > \bar{S}_3$ and the rotation at the hinge A terminates earlier than that at the hinge B. The instant $t = t_A$ at which $\theta(t_A) = 0$ is determined from (54), namely

$$t_A = t_1\left(1 - \frac{S_1}{\bar{S}_3}\right) \tag{55}$$

and there is $\bar{S}_3 < 0$ since $S_1 > 0$ and $t_A > t_1$. From the second relation (54) and from (55), we obtain

$$\dot{\psi}(t_A) = Nt_1S_1\left(1 - \frac{\bar{T}_3}{\bar{S}_3}\right) \tag{56}$$

and since $\bar{T}_3 > \bar{S}_3$ and $\dot{\psi}(t_A) > 0$, there must be $\bar{T}_3 > 0$. The values of $\theta(t_A)$ and $\psi(t_A)$ are expressed as follows

$$\theta(t_A) = \frac{1}{2}Nt_1{}^2S_1\left(1 - \frac{S_1}{\bar{S}_3}\right), \qquad \psi(t_A) = \frac{1}{2}Nt_1{}^2S_1\left(1 - 2\frac{S_1}{\bar{S}_1} + \frac{\bar{T}_3T_1}{\bar{S}_3{}^2}\right) \tag{57}$$

During the third phase $t_A \le t \le t_f$, there is $\ddot{\theta} = 0$, $\ddot{\psi} = N\bar{T}_2$ and

$$\dot{\theta}(t) = 0, \quad \dot{\psi}(t) = Nt_1S_1\left(1 - \frac{\bar{T}_3}{\bar{S}_3}\right) + N\bar{T}_2(t - t_A), \tag{58}$$

where $\bar{T}_2$ is determined from the equality (48″). The motion terminates at the instant

$$t_f = t_A - t_1\frac{S_1}{\bar{T}_2}\left(1 - \frac{\bar{T}_3}{\bar{S}_3}\right) \tag{59}$$

and the final values of the angles θ and ψ are

$$\theta(t_f) = \theta(t_A), \quad \psi(t_B) = \frac{1}{2}Nt_1{}^2S_1\left[1 - 2\frac{S_1}{\bar{S}_3} + \frac{\bar{T}_3S_1}{\bar{S}_3^2} - \frac{S_1}{\bar{T}_2}\left(1 - \frac{\bar{T}_3}{\bar{S}_3}\right)^2\right]. \tag{60}$$

The deflection at A is expressed as follows

$$u(0, t_f) = \frac{1}{2}NLt_1{}^2W_2, \quad W_2 = (1 - \alpha)S_1\left[1 - 2\frac{S_1}{\bar{S}_3} + \frac{\bar{T}_3S_1}{\bar{S}_3^2} - \frac{S_1}{\bar{T}_2}\left(1 - \frac{\bar{T}_3}{\bar{S}_3}\right\}^2\right] + \alpha S_1\left(1 + \frac{S_1}{\bar{S}_3}\right). \tag{61}$$

Let us state the conditions under which this case is valid. First, we have $S_1 > 0$, $\bar{S}_3 < 0$, $\bar{T}_3 > 0$. Secondly, the strong inequality (48″) must be fulfilled for $S = S_1$, $T = S_1$, $P \neq 0$ and the strong inequality (48′) should hold for $S = 0$, $T = \bar{T}_3$, $P = 0$.

The solutions for other cases can be obtained analogously and the details will not be discussed here. We present only final expressions for deflection at the beam center.

Case 3 (2–2). The initial yield mechanism 2 occurs during the whole period of motion $0 \le t \le t_f$. The non-dimensional central deflection is

$$W_3 = (1 - \alpha)T_2\left(1 - \frac{T_2}{\bar{T}_2}\right). \tag{62}$$

This case occurs when $T_2 > 0$ and the inequality (48′) holds for $S = 0$, $T = T_2$, $P \neq 0$ and for $S = 0$, $T = \bar{T}_2$, $P = 0$.

Case 4 (2-3-1). Now, the non-dimensional final deflection equals

$$W_4 = \alpha\frac{\bar{S}_3T_2{}^2}{(\bar{S}_3 - \bar{T}_3)^2}\left(1 - \frac{\bar{S}_3}{\bar{S}_1}\right) + (1 - \alpha)T_2\left[1 + \frac{2T_2}{\bar{S}_3 - \bar{T}_3} + \frac{T_2}{(\bar{S}_3 - \bar{T}_3)^2}\left(\bar{T}_3 - \frac{\bar{S}_3{}^2}{\bar{S}_1}\right)\right]. \tag{63}$$

The following conditions must now be fulfilled: (i) $T_2 > 0$, $\bar{S}_3 > \bar{T}_3$, $\bar{S}_3 > 0$ (ii) the inequality (48′) holds for $S = 0$, $T = T_2$, $P \neq 0$ and (iii) the inequality (48″) occurs for $S = \bar{S}_1$, $T = \bar{S}_1$, $P = 0$.

Case 5 (3–3–1). The non-dimensional deflection at the beam center is now expressed as follows

$$W_5 = [(1-\alpha)T_3 + \alpha S_3]\left(1 - 2\frac{T_3 - S_3}{\bar{T}_3 - \bar{S}_3}\right) + [(1-\alpha)\bar{T}_3 + \alpha\bar{S}_3]\left(\frac{T_3 - S_3}{\bar{T}_3 - \bar{S}_3}\right)^2 - \frac{1}{\bar{S}_1}\left(\frac{S_3\bar{T}_3 - \bar{S}_3 T_3}{\bar{T}_3 - \bar{S}_3}\right)^2. \tag{64}$$

The conditions for occurrence of this case are as follows: (i) $T_3 > S_3 > 0$, $\bar{S}_3 > \bar{T}_3$, (ii) the inequality (48″) holds for $S = \bar{S}_1$, $T = \bar{S}_1$, $P = 0$.

Case 6 (3–3–2). Now we have

$$W_6 = (1-\alpha)\left[T_3\left(1 - 2\frac{S_3}{\bar{S}_3}\right) + \bar{T}_3\left(\frac{S_3}{\bar{S}_3}\right)^2 - \frac{1}{\bar{T}_2}\left(\frac{T_3\bar{S}_3 - \bar{T}_3 S_3}{\bar{S}_3}\right)^2\right] + \alpha S_3\left(1 - \frac{S_3}{\bar{S}_3}\right). \tag{65}$$

This case occurs when (i) $T_3 > S_3 > 0$, $\bar{S}_3 < \bar{T}_3$, $\bar{S}_3 < 0$, (ii) the inequality (48′) is valid for $S = 0$, $T = \bar{T}_2$, $P = 0$.

6.2. *Numerical results*

The analytical expressions for final deflection at $x = 0$ and inequalities for particular six cases constitute the basis for numerical computations. The numerical scheme is as follows: (i) for a set of parameters α, γ, P we first calculate the quantities S_i, T_i and $\bar{S}_i$, $\bar{T}_i$ $(i = 1, 2, 3)$, (ii) next the proper case for which all inequalities are satisfied is singled out and the non-dimensionsl deflection W_i $(i = 1, 2, 3, \ldots 6)$ occuring in the formula for final deflection at the center is determined; then

$$u_i(0, t_f) = \frac{1}{2}Nlt_1^2 W_i. \tag{66}$$

Next, (iii) the range of validity of the presented solution is verified.

In order to verify (iii), it must be remembered that the solution presented in this work is valid only for moderate pressures when the assumption on stationary plastic hinges at A and B is valid. This assumption is satisfied when the bending moment is a monotonically decreasing function from the maximum value at $x = 0$ to zero at $x = l$. This condition will be fulfilled when $d^2M/dx^2 < 0$ for $x = 0$. Using non-dimensional quantities, we express this condition in the form

$$3\gamma[\alpha S + (1-\alpha)T] - P\Delta \leq 0. \tag{67}$$

The validity of this inequality must be checked for the first and the second phase only since during the third phase there is $S \leq 0$, $T \leq 0$, $P = 0$ and (67) is valid. This inequality is violated for sufficiently high pressure P: thus our solution is valid for the range of moderate pressures.

Computations were carried out for $P = 2$ and $P = 3$ and occurence of different yield mechanisms during motion is illustrated for $P = 3$ in Fig. 12. The numbers in the plane (α, γ) show which case of motion occurs at a given point. The level lines of W_i are also plotted in this figure. We note that cases 2 and 6 do not occur within considered ranges of α and γ. Moreover, one important conclusion can be drawn, namely that the motion is terminated through the one-degree-of freedom mechanism 1 or 2 and the mechanism 3 corresponds only to the transient period. Thus the mechanism 3 is an *unstable mode* which passes into stable modes 1 or 2. The present analysis thus casts more light on stability of modes which was only briefly mentioned in considering the case of impulsive loading. In fact, using mode solution, we were not able to study stability of modes since this would require the analysis of perturbed motions with respect to mode solutions.

It is also interesting to compare the regions of occurence of final modes 1 and 2. In Fig. 12 these regions are separated by a line drawn through points separating cases 3 and 4. When compared with the line separating modes 1 and 2 in Fig. 3, it is seen that these two lines lie fairly closely, see Fig. 13. Thus the present analysis complements the previous conclusions on mode separation and their stability. The optimal solution is marked by an asterisk in Fig. 12.

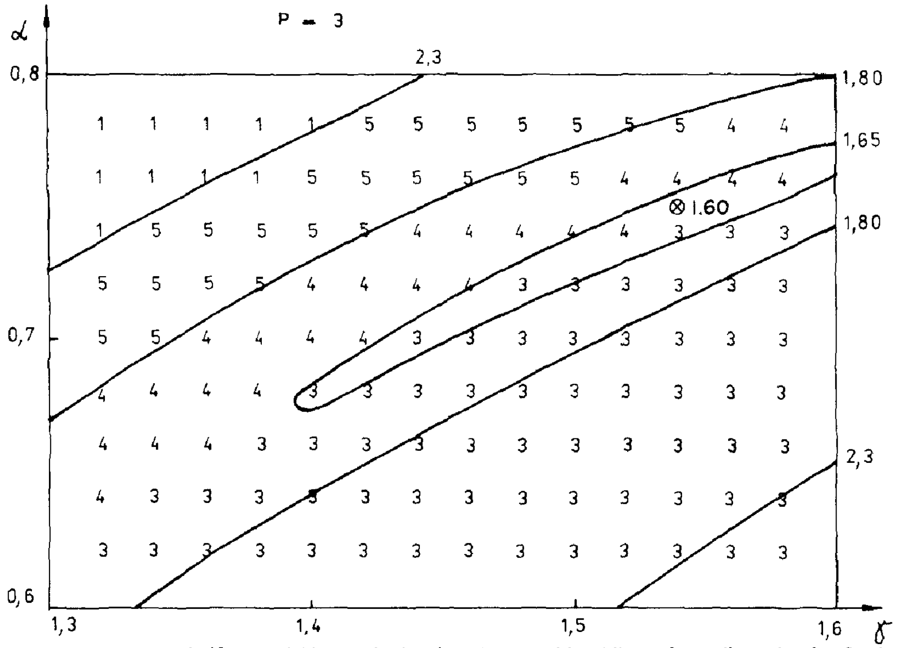


Fig. 12. Occurence of different yield cases in the plane (α, γ) and level lines of non-dimensional deflection W_i. Optimal design is marked by ⊗.

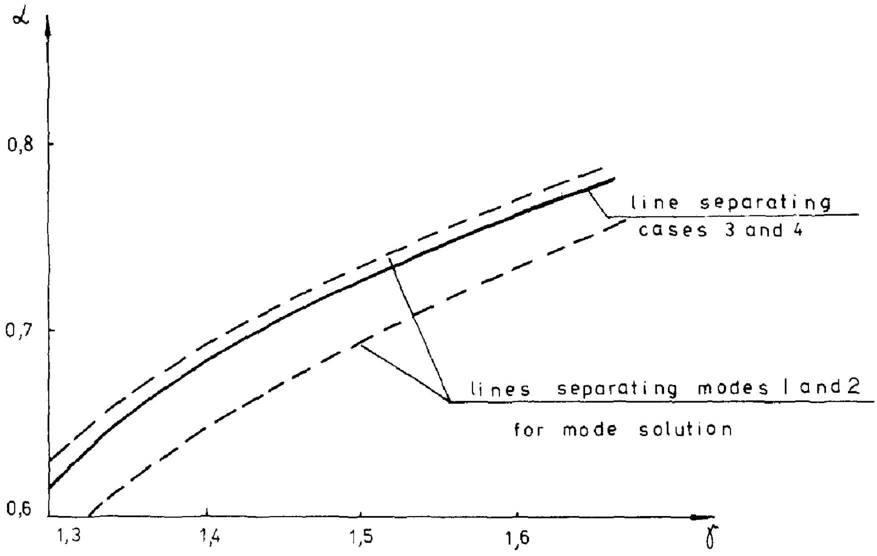


Fig. 13. Lines separating final modes 1 and 2 for impulsive and pressure loading.

The minimal non-dimensional deflection for $P = 3$ is $W(3) = 1.60$ and occurs for $\alpha = 0.75$, $\gamma = 1.54$; for $P = 2$ we would have respectively $W(2) = 0.47$ $\alpha = 0.74$, $\gamma = 1.51$. For a beam of constant thickness, we have

$$Wu(P) = \frac{1}{2}(P-1)P \tag{68}$$

and

$$\frac{W(3)}{Wu(3)} = 0.53, \qquad \frac{W(2)}{Wu(2)} = 0.49. \tag{69}$$

Thus for a beam of constant volume the deflection at the center is twice as large when using the uniform thickness design instead of two-segment design. This result indicate that damping capacity of a structure may be considerably increased by using piecewise-constant cross sections.

7. CONCLUDING REMARKS

The presented analysis for the case of pressure loading indicates that significant complexity may arise when analysing the effect of configuration or dimension changes of a structure on the mechanism of motion. In fact, for a three-segment beam the analysis would be very complex. Therefore simplified models, such as for instance, mode solution should be of great help in considering synthesis problems under dynamic loads. The application of mode solutions to optimization of non-linear elastic or viscous structures was presented in [6].

Although this paper does not propose a general theory, it nevertheless casts a light on some interesting aspects of dynamic behaviour, such as (i) non-uniqueness of modes for rigid-plastic structures, (ii) stability and instability of some modes, (iii) importance of one-degree-of freedom modes in design. It seems therefore that study of interaction between design and behaviour of inelastic structures under dynamic loads may become important and fruitful field of research, especially in exploring their optimal energy damping capacities.

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